

Thermodynamics of Chaplygin gas

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Abstract

We clarify thermodynamics of the Chaplygin gas by introducing the integrability condition. All thermal quantities are derived as functions of either volume or temperature. Importantly, we find a new general equation of state, describing the Chaplygin gas completely. We confirm that the Chaplygin gas could show a unified picture of dark matter and energy which cools down through the universe expansion without any critical point (phase transition).

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I. INTRODUCTION

We start with an exotic perfect fluid, named Chaplygin gas which obeys the following adiabatic equation of state [1, 2]

$$p = -\frac{A}{\rho^\alpha}, \quad (1)$$

where ρ is the energy density of the fluid defined by $\rho = E/V$ and α is constant and positive: $\alpha > 0$. In this work, we choose $\alpha = 1$ as the Chaplygin gas for simplicity. The parameter A is positive and considered as a universal constant. Here we choose $A = 1$ for numerical computations. This gas behaves as pressureless gas (dust) at high energy densities, while it behaves a cosmological constant with negative pressure at low energy densities. Hence the Chaplygin gas is regarded as a unified model of dark matter and dark energy [3, 4, 5].

For the thermodynamic study [6, 7], we apply the combination of the first- and second-law of thermodynamics to the system with volume V . Then it leads to

$$TdS = d(\rho V) + pdV = d[(\rho + p)V] - Vdp. \quad (2)$$

The integrability condition is necessary to define the Chaplygin gas as a thermodynamic system [8, 9, 10]. It is given by

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T} \quad (3)$$

which leads to the relation between the pressure (energy density) and temperature

$$dp = \frac{\rho + p}{T} dT. \quad (4)$$

Plugging Eq.(4) into Eq.(2), we have the differential relation,

$$dS = \frac{1}{T} d[(\rho + p)V] - (\rho + p)V \frac{dT}{T^2} = d \left[\frac{(\rho + p)V}{T} + C \right] \quad (5)$$

where C is a constant. The entropy is defined by

$$S \equiv \frac{(\rho + p)V}{T} \quad (6)$$

up to an additive constant. Even for an adiabatic process of $S = \text{const}$, the same definition of entropy follows from the conservation law which can be rewritten as

$$d[(\rho + p)V] = Vdp. \quad (7)$$

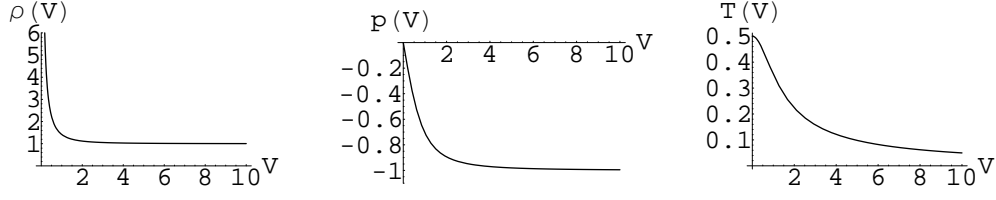


FIG. 1: Graphs of energy density ρ , pressure p , and temperature as functions of V with $A = B = 1$ and $S = 2$.

Inserting the integrability condition Eq.(4) into Eq.(7), one recovers Eq.(6) immediately. Hence we use the equation (6) as a defining equation of the temperature for an adiabatic process [12, 13]

$$T \equiv \frac{(\rho + p)}{S} V. \quad (8)$$

On the other hand, the conservation law (7) plays an important role in the homogeneous and isotropic FRW universe which is described by two Friedmann equations based on the Robertson-Walker metric

$$H^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2}, \quad (9)$$

$$\dot{H} = -4\pi G(\rho + p) + \frac{k}{a^2} \quad (10)$$

where $H = \dot{a}/a$ is the Hubble parameter and $k = -1, 0, 1$ represent the three-dimensional space with the negative, zero, and positive spatial curvature, respectively. This could be derived from Eqs.(9) and (10) as

$$\dot{\rho} + 3H(\rho + p) = 0 \quad (11)$$

Hence, one equation among three is redundant. Here we may choose the first Friedmann equation (9) and the conservation law (11) as two relevant equations. Importantly, Eq.(11) together with Eq.(1) is solved to give the energy density

$$\rho(a) = \sqrt{A + \frac{B}{a^6}}, \quad (12)$$

where B is an integration constant. By choosing a positive value for B , we see that for small a ($a^6 \ll B/A$), Eq.(12) and the pressure take the forms approximately

$$\rho \sim \frac{\sqrt{B}}{a^3}, \quad p \sim 0 \quad (13)$$

which corresponds to a matter-dominated universe. For a large value a , it follows that

$$\rho \sim \sqrt{A}, \quad p \sim -\sqrt{A} \quad (14)$$

which corresponds to a dark energy-dominated universe.

Taking $V = a^3$, we have the relations

$$\rho(V) = \sqrt{A + \frac{B}{V^2}}, \quad p(V) = -\frac{A}{\sqrt{A + \frac{B}{V^2}}}. \quad (15)$$

Considering Eq.(8), one has the temperature

$$T(V) = \frac{1}{S\sqrt{AV^2 + B}}. \quad (16)$$

Furthermore, the equation of state $\omega(V)$ and squared speed of sound $v^2(V)$ [11] is given by

$$\omega(V) \equiv \frac{p}{\rho} = -\frac{A}{A + \frac{B}{V^2}}, \quad v^2(V) \equiv \frac{\partial p}{\partial \rho} = \frac{A}{A + \frac{B}{V^2}}. \quad (17)$$

In this case there is no restriction on thermodynamic quantities as functions of V ($0 \leq V \leq \infty$). As is shown in Fig. 1, we recover the matter-dominated universe of $\rho \rightarrow \infty$, $p \rightarrow 0$ as $V \rightarrow 0$, while the dark-energy dominated universe of $\rho \rightarrow 1$, $p \rightarrow -1$ appears as $V \rightarrow \infty$. Also we note that the temperature goes zero only when the volume goes to infinity, showing that the third-law of thermodynamics is satisfied with the Chaplygin gas, contrast to Ref.[6].

II. THERMODYNAMICS

From now on, we wish to derive two important relations from the integrability condition Eq.(4). We have one connection for the pressure and temperature

$$\frac{p \, dp}{p^2 - A} = \frac{dT}{T} \quad (18)$$

which leads to

$$\ln \left[\frac{p^2 - A}{-A} \right] = \ln \left[\frac{T^2}{T_*^2} \right]. \quad (19)$$

Here T_* is the temperature corresponding to $p = 0$. In this case, we obtain the relation between pressure and temperature [6]

$$p(T) = -\sqrt{A} \sqrt{1 - \frac{T^2}{T_*^2}}. \quad (20)$$

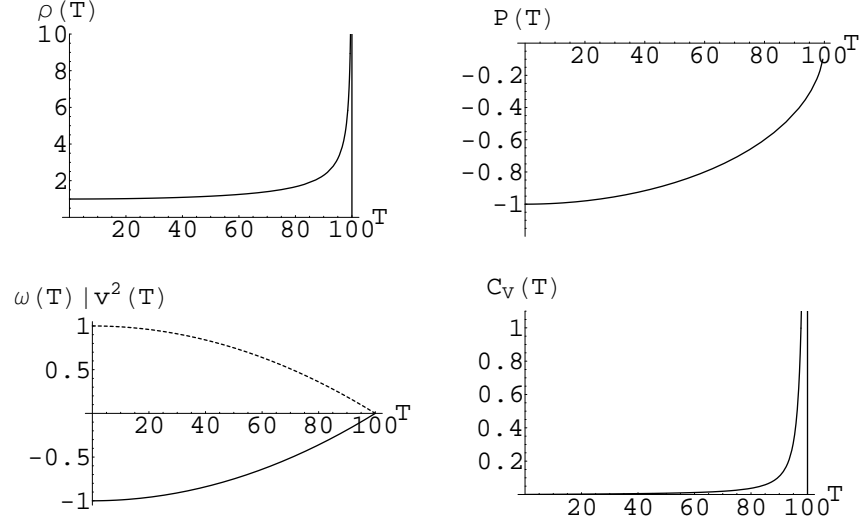


FIG. 2: Graphs of energy density $\rho(T)$, pressure $p(T)$, equation of state $\omega(T)$ (solid)| squared speed of sound $v^2(T)$ (dotted), and heat capacity $C_V(T)$ with $A = 1$ and $V = 1$. All these show that a process from $T = T^* = 100$ to $T = 0$ is a thermodynamically stable transition.

Also, from the other form of integrability condition

$$\frac{Ad\rho}{\rho(\rho^2 - A)} = \frac{dT}{T}, \quad (21)$$

we derive the important relation [6]

$$\rho(T) = \frac{\sqrt{A}}{\sqrt{1 - \frac{T^2}{T_*^2}}}. \quad (22)$$

Here we observe the allowed range of temperature: $0 \leq T \leq T_*$. Using these expressions, one has two quantities of equation of state and squared speed of sound as function of T

$$\omega(T) = -\frac{A}{\rho^2(T)} = -1 + \frac{T^2}{T_*^2}, \quad v^2(T) = \frac{\partial p}{\partial \rho} = 1 - \frac{T^2}{T_*^2} \quad (23)$$

which are important to describe a fluid of the Chaplygin gas.

Finally, the heat capacity is calculated to be

$$C_V(T) = V \frac{\partial \rho}{\partial T} = \frac{\sqrt{AV}}{T_*^2} \frac{T}{\left[1 - \frac{T^2}{T_*^2}\right]^{3/2}}. \quad (24)$$

All pictures of thermodynamic quantities are depicted as functions of temperature in Fig. 3. These show that an evolution from $T = T^* = 100$ to $T = 0$ is a thermodynamically stable

transition without any critical point. That is, even though the Chaplygin gas provides an accelerating universe, its fluid has a stable nature of positive squared sound velocity and positive heat capacity during the evolution.

In this section we recover all known thermodynamic quantities as functions of temperature using the integrability condition .

III. GENERAL EQUATION OF STATE

Combining Eq.(8) with Eq.(1), we have the second-order equation for pressure p

$$Vp^2 - TS p - AV = 0, \quad (25)$$

whose solution leads to a complete equation of state for p, V, T

$$pV = T \left[\frac{S}{2} - \sqrt{A \frac{V^2}{T^2} + \frac{S^2}{4}} \right]. \quad (26)$$

For the case of $S = 2$, it takes a simpler form

$$pV = T \left[1 - \sqrt{1 + A \frac{V^2}{T^2}} \right]. \quad (27)$$

As far as we know, this is the first equation of state for an adiabatic Chaplygin gas. For an ideal gas, we have $pV^\gamma = \text{const}$ with $\gamma = C_p/C_V$.

On the other hand, we have the second-order equation for energy density ρ

$$V\rho^2 - TS\rho - AV = 0, \quad (28)$$

whose solution leads to

$$\rho V = T \left[\frac{S}{2} + \sqrt{A \frac{V^2}{T^2} + \frac{S^2}{4}} \right]. \quad (29)$$

For the case of $S = 2$, it takes a simpler form

$$\rho V = T \left[1 + \sqrt{1 + A \frac{V^2}{T^2}} \right]. \quad (30)$$

We check that adding (26) and (29) leads to (8). Also dividing (26) by (29) leads to the equation of state as functions of T and V

$$\omega(T, V) = -\frac{A}{\rho(T, V)}. \quad (31)$$

We consider the case of a constant temperature (See Figs. 1 and 3). In the limit of $V \rightarrow \infty$, one has

$$\rho \rightarrow \sqrt{A}, \quad p \rightarrow -\sqrt{A}, \quad (32)$$

while in the limit of $V \rightarrow 0$, one finds

$$\rho \rightarrow \frac{2T}{V}(=\infty), \quad p \rightarrow 0. \quad (33)$$

Comparing Fig. 3 with Fig. 1, we find that Eqs.(27) and (30) could describe an adiabatic, isothermal process of the Chaplygin gas well.

For the case of constant volume (See Figs. 2 and 3), one has

$$\rho \rightarrow \sqrt{A}, \quad p \rightarrow -\sqrt{A} \quad (34)$$

in the limit of $T \rightarrow 0$, while in the limit of $T \rightarrow \infty$, one finds

$$\rho \rightarrow \frac{2T}{V}(=\infty), \quad p \rightarrow 0. \quad (35)$$

The former describes the dark energy-dominated universe, while the latter describes the matter-dominated universe. We find that Eqs.(27) and (30) could describe an adiabatic, isochoric process of the Chaplygin gas well.

IV. DISCUSSIONS

We clarify thermodynamics of the Chaplygin gas by introducing the integrability condition Eq. (3) and the temperature of (8). All thermal quantities are derived as functions of either temperature or volume. In this case, we show that the third-law of thermodynamics is satisfied with the Chaplygin gas. Furthermore, we find a new general equation of state, describing the Chaplygin gas as function of temperature and volume completely. For the generalized Chaplygin gas with $\alpha > 0$, we expect to have similar behaviors as the Chaplygin gas did show.

Consequently, we confirm that the Chaplygin gas could show a unified picture of dark matter and energy which cools down through the universe expansion without any critical point.

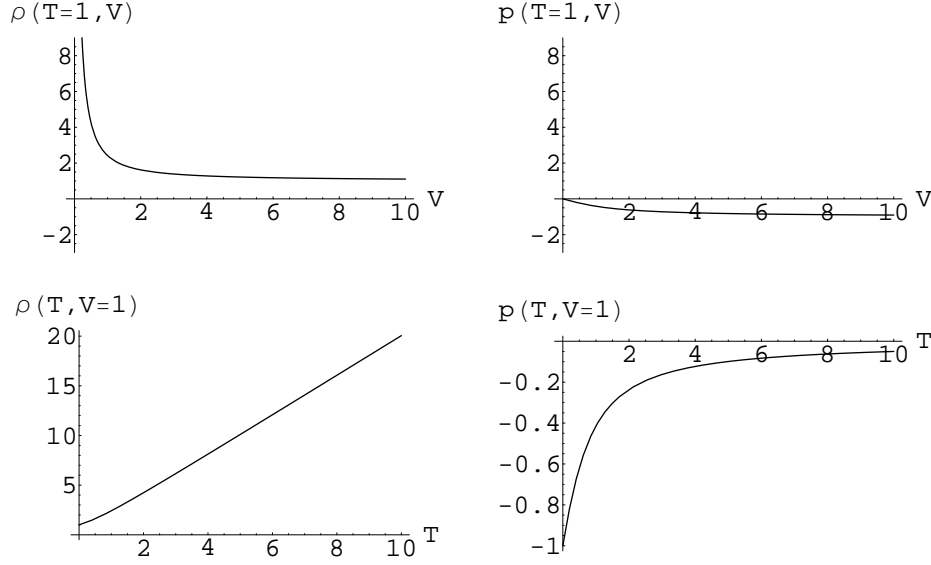


FIG. 3: Graphs of $\rho(T = 1, V)$, $p(T = 1, V)$, $\rho(T, V = 1)$, and $p(T, V = 1)$ with $A = 1$ and $S = 2$. These mimics Fig. 1 and Fig. 2.

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